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## Vector spaces

A **vector space** consists of an abstract set of elements, called **vectors** that can be added together and scaled. The notion of “scaling” is modeled by a **field**. That is, a vector space involves a set of vectors  $\mathcal{V}$  and a field of scalars  $\mathcal{F}$  for which one can add together vectors in  $\mathcal{V}$  as well as scale these vectors by elements in the field  $\mathcal{F}$  according to the set of rules outlined in Definition 1. Axioms 1-5 of the definition describe how vectors can be added together. Axioms 6-10 describe how these vectors can be scaled using the field of scalars.

**Definition 1** Given a set of objects  $\mathcal{V}$  called vectors and a field  $\mathcal{F} := (C, +, \cdot, -, ^{-1}, 0, 1)$  where  $C$  is the set of elements in the field, called scalars, the tuple  $(\mathcal{V}, \mathcal{F})$  is a **vector space** if for all  $\mathbf{v}, \mathbf{u}, \mathbf{w} \in \mathcal{V}$  and  $c, d \in C$ , it obeys the following ten axioms:

1.  $\mathbf{u} + \mathbf{v} \in \mathcal{V}$
2.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3.  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There exists a zero vector  $\mathbf{0} \in \mathcal{V}$  such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. For each  $\mathbf{u} \in \mathcal{V}$  there exists a  $\mathbf{u}' \in \mathcal{V}$  such that  $\mathbf{u} + \mathbf{u}' = \mathbf{0}$ . We call  $\mathbf{u}'$  the negative of  $\mathbf{u}$  and denote it as  $-\mathbf{u}$
6. The scalar multiple of  $\mathbf{u}$  by  $c$ , denoted by  $c\mathbf{u}$  is in  $\mathcal{V}$
7.  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8.  $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9.  $c(d\mathbf{u}) = (cd)\mathbf{u}$
10.  $1\mathbf{u} = \mathbf{u}$

## Vector subspaces

A vector space can be induced by an appropriate subset of vectors from some larger vector space. We call such a subspace a **vector subspace** (Definition 2). By merits of the original vector space, seven out of 10 axioms will always hold; however, there are

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three axioms that may not hold that must be verified whenever a subset of vectors from a vector space are to be considered when formulating a vector subspace.

**Definition 2** A subset of vectors  $\mathcal{H} \subseteq \mathcal{V}$  from a vector space  $(\mathcal{V}, \mathcal{F})$  forms a **vector subspace**  $(\mathcal{H}, \mathcal{F})$  if the following three properties hold:

1. The zero vector of  $\mathcal{V}$  is in  $\mathcal{H}$ .
2.  $\mathcal{H}$  is closed under vector addition. That is, for each  $\mathbf{u}, \mathbf{v} \in \mathcal{H}$ , the vector  $\mathbf{u} + \mathbf{v}$  is also in  $\mathcal{H}$ .
3.  $\mathcal{H}$  is closed under scalar multiplication. That is, for each  $\mathbf{u} \in \mathcal{H}$  and each scalar  $c \in \mathcal{F}$ , the vector  $c\mathbf{u}$  is in  $\mathcal{H}$ .

## Properties

1. **The zero vector is unique** (Theorem 1). There is only one distinct zero vector in a vector space.
2. **Any vector multiplied by the zero scalar is the zero vector** (Theorem 2). The zero scalar converts any vector into the zero vector. That is, given a vector  $\mathbf{v}$ , it holds that

$$0\mathbf{v} = \mathbf{0}$$

This is analogous to how any number multiplied by zero becomes zero.

3. **The negative of a vector is unique** (Theorem 3). Given a vector  $\mathbf{v}$ , we denote its negative vector as  $-\mathbf{v}$ . This is analogous to each number  $x$  having a matching negative number  $-x$  that lies  $|x|$  distance from 0 on the opposite side of 0.
4. **Multiplying a negative vector by the scalar -1 produces its negative vector** (Theorem 4). That is, given a vector  $\mathbf{v}$

$$-1\mathbf{v} = -\mathbf{v}$$

This is analogous to the fact that if you multiply any number  $x$  by  $-1$  you get the number  $-x$  that lies  $|x|$  distance from 0 on the opposite side of 0.

5. **The zero vector multiplied by any scalar is the zero vector** (Theorem 5). The zero vector remains the zero vector despite being multiplied by any scalar. That is,

$$c\mathbf{0} = \mathbf{0}$$

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for any  $c \in \mathcal{F}$ . This is analogous to the fact that zero multiplied by any number remains zero.

6. **The only vector whose negative is not distinct from itself is the zero vector** (Theorem 6). For every vector other than the zero vector, its negative vector is a distinct vector in the vector space. For the zero vector, its negative is itself. This is analogous to the fact that for any number  $x \neq 0$ , the number  $-x$  is a distinct number from  $x$  that lies on the opposite side of 0. However, for  $x = 0$ ,  $-x = x$ .

**Theorem 1** *Given vector space  $(\mathcal{V}, \mathcal{F})$ , the zero vector is unique.*

**Proof:** Assume for the sake of contradiction that there exists a vector  $\mathbf{a}$  such that  $\mathbf{a} \neq \mathbf{0}$  and that  $\forall \mathbf{v} \in \mathcal{V}$

$$\mathbf{a} + \mathbf{v} = \mathbf{v}$$

Then, this implies that :

$$\mathbf{a} + \mathbf{0} = \mathbf{0}$$

However, axiom 4 states that for the zero-vector

$$\mathbf{a} + \mathbf{0} = \mathbf{a}$$

Since  $\mathbf{a} \neq \mathbf{0}$ , we reach a contradiction. Therefore, there does not exist a vector  $\mathbf{a} \neq \mathbf{0}$  for which  $\forall \mathbf{v} \in \mathcal{V}$   $\mathbf{a} + \mathbf{v} = \mathbf{v}$ . Thus, the zero-vector is unique.

□

**Theorem 2** *Given a vector space  $(\mathcal{V}, \mathcal{F})$*

$$\forall \mathbf{v} \in \mathcal{V}, 0\mathbf{v} = \mathbf{0}$$

**Proof:** Assume for the sake of contradiction that there exists a vector  $\mathbf{a} \neq \mathbf{0}$  such that

$$0\mathbf{v} = \mathbf{a}$$

Now, for any scalar  $c \neq 0$ , we have

$$\begin{aligned} c\mathbf{v} &= (c + 0)\mathbf{v} \\ &= c\mathbf{v} + 0\mathbf{v} && \text{by axiom 8} \\ &= c\mathbf{v} + \mathbf{a} \end{aligned}$$

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Our assumption assumed that  $\mathbf{a} \neq \mathbf{0}$  must be false because by Theorem 1 the only vector  $\mathbf{a}$  for which  $c\mathbf{v} + \mathbf{a} = c\mathbf{v}$  would be true is the zero-vector.

□

**Theorem 3** Given a vector space  $(\mathcal{V}, \mathcal{F})$  and vector  $\mathbf{v} \in \mathcal{V}$ , its negative,  $-\mathbf{v}$ , is unique. That is,

$$\mathbf{v} + \mathbf{a} = \mathbf{0} \iff \mathbf{a} = -\mathbf{v}$$

**Proof:**

We need only prove  $\mathbf{v} + \mathbf{a} = \mathbf{0} \implies \mathbf{a} = -\mathbf{v}$ . The other direction is stated in the axioms.

$$\begin{aligned} \mathbf{v} + \mathbf{a} &= \mathbf{0} \\ \implies -\mathbf{v} + \mathbf{v} + \mathbf{a} &= -\mathbf{v} + \mathbf{0} \\ \implies [-\mathbf{v} + \mathbf{v}] + \mathbf{a} &= -\mathbf{v} \\ \implies \mathbf{0} + \mathbf{a} &= -\mathbf{v} && \text{by axiom 5} \\ \implies \mathbf{a} &= -\mathbf{v} && \text{by axiom 4} \end{aligned}$$

□

**Theorem 4** Given a vector  $\mathbf{v} \in \mathcal{V}$ , its negative is  $(-1)\mathbf{v}$ . That is,

$$-\mathbf{v} = (-1)\mathbf{v}$$

**Proof:**

$$\begin{aligned} \mathbf{v} + (-1)\mathbf{v} &= (1)\mathbf{v} + (-1)\mathbf{v} && \text{by axiom 10} \\ &= (1 - 1)\mathbf{v} && \text{by axiom 8} \\ &= 0\mathbf{v} \\ &= \mathbf{0} && \text{by Theorem 2} \end{aligned}$$

Then, by axiom 5, it must be that  $(-1)\mathbf{v} = -\mathbf{v}$ .

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**Theorem 5** Given a vector space  $(\mathcal{V}, \mathcal{F})$

$$c\mathbf{0} = \mathbf{0} \iff \mathbf{a} = \mathbf{0}$$

**Proof:**

$$\begin{aligned} \mathbf{0} + \mathbf{0} &= \mathbf{0} && \text{by axiom 4} \\ c(\mathbf{0} + \mathbf{0}) &= c\mathbf{0} \\ c\mathbf{0} + c\mathbf{0} &= c\mathbf{0} && \text{by axiom 8} \end{aligned}$$

By Theorem 1, the only vector  $\mathbf{a}$  in  $\mathcal{V}$  for which  $\mathbf{a} + \mathbf{v} = \mathbf{v}$  for all vectors  $\mathbf{v} \in \mathcal{V}$  is the zero vector  $\mathbf{0}$ . Thus,  $c\mathbf{0} = \mathbf{0}$ .

□

**Theorem 6** Given a vector space  $(\mathcal{V}, \mathcal{F})$

$$-\mathbf{0} = \mathbf{0}$$

**Proof:**

$$\begin{aligned} \mathbf{a} + -\mathbf{a} &= \mathbf{0} && \text{by axiom 5} \\ \mathbf{a} + \mathbf{a} &= \mathbf{0} && \text{assume } \mathbf{a} = -\mathbf{a} \\ \implies 2\mathbf{a} &= \mathbf{0} \\ \implies \mathbf{a} &= \mathbf{0} && \text{by Theorem 5} \end{aligned}$$

Thus, if we assume  $\mathbf{a} = -\mathbf{a}$ , then  $\mathbf{a}$  must be the zero vector.

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