The binomial theorem

The **binomial Theorem** provides an alternative form of a binomial expression raised to a power:

Theorem 1

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k}$$

Proof:

We first begin with the following polynomial:

$$(a + b)(c + d)(e + f)$$

To expand this polynomial we iteratively use the distribut.ive property. For example, the first step in the expansion is

$$(a + b)(c + d)(e + f) = a(c + d)(e + f) + b(c + d)(e + f)$$

. Notice that as a result, *either* the factor a *or* the factor b will appear in each term of the fully expanded polynomial. In fact, for each binomial factor (i.e. for each of $(a + b)$, $(c + d)$, and $(e + f)$), only one of the terms of the binomial will appear in a given term of the fully expanded polynomial. We see that this holds in this example:

$$(a + b)(c + d)(e + f) = ace + ace + acf + ade + adf + bce + bcf + bde + bdf$$

. We imagine the process of "picking" one term from each binomial in order to form a term in the fully expanded polynomial. In fact, *every* possible term we can form by picking one value from a distinct binomial expression is represented in the fully expanded polynomial.

Now, let's look at the following polynomial:

$$(x + y)^3 = (x + y)(x + y)(x + y) = xxx + xyx + yxx + yyx + xxy + xyy + yxy + yyy$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

. Let's say we're interested in all terms in the expanded polynomial that have k of the x values. By the previous observation, the term that has k of the x values must have $n - k$ of the y values because we only pick a single value from each binomial expression – in this case, either an x or a y. How many of the terms in the expanded polynomial will have k of the x values? Every combination of ways of picking k of the x values from the binomial factors will result in a term of the form $x^k y^{n-k}$ in the expanded polynomial. Thus, there will be $\binom{n}{k}$ such terms.

Finally, there are terms in the polynomial with k values of x for every value of k between 0 and n. This is a result of the fact that every combination of terms where each term is picked from a single binomial factor is represented. This final observation leads to the conclusion of the binomial theorem:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k}$$

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