Normed vector spaces

A **normed vector space** is a vector space where each vector is associated with a "length". In the 2 or 3 dimensional Euclidean vector space, this notion is intuitive: the norm of a vector can simply be defined to be the length of the arrow. However, the concept of a norm generalizes this idea of the length of an arrow vector. Moreover, the norm of a vector changes when we multiply a vector by a scalar. Thus, a norm implements our intuition that scaling a vector changes its length.

Definition 1 *More rigorously, a normed vector space is a tuple* $(\mathcal{V}, \mathcal{F}, ||.||)$ *where* $\mathcal{V}$ *is a vector space,* $\mathcal{F}$ *is a field, and* $||.||$ *is a function called the* ***norm*** *that maps vectors to strictly positive lengths:*

$$||.|| : \mathcal{V} \to [0, \infty)$$

The norm function must satisfy the following conditions:

1. $\forall \mathbf{v} \in \mathcal{V} \quad ||\mathbf{v}|| \geq 0$
2. $||\alpha\mathbf{x}|| = |\alpha| \, ||\mathbf{x}||$
3. $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$

Intuition

The norm of a vector captures the a notion of "length" for a vector. Below we outline the intuition behind each axiom in Definition 1 and describe how these axioms capture the notion of "length":

1. says that all vectors should have a positive length. This enforces our intuition that a "length" is a positive quantity.
2. says that if we multiply a vector by a scalar, it's length should increase by the magnitude (i.e. the absolute) value of that scalar. This axiom ties together the notion of scaling vectors (Axiom 6 in the definition of a vector space) to the notion of "length" for a vector. It essentially says that to scale a vector is to stretch the vector.
3. says that the length of the sum of two vectors should not exceed the sum of the lengths of each vector. This enforces our intuition that if we add together two

objects that each have a "length", the resultant object should not exceed the sum of the lengths of the original objects.

Properties

Only the zero vector has zero length

From the definition of the normed vector space, it can be proven that only the zero vector has zero length.

Theorem 1

$$||\mathbf{a}|| = 0 \iff \mathbf{a} = \mathbf{0}$$

Proof:

$$\begin{aligned} ||\mathbf{0}|| &= ||0\mathbf{a}|| && \text{for any vector } \mathbf{a} \in \mathcal{V} \\ &= |0|\, ||\mathbf{a}|| && \text{by Axiom 2} \\ &= 0 \end{aligned}$$

□

Unit Vectors

In a normed vector space, a unit vector is a vector with norm equal to 1 (Definition 2). Given a vector $\mathbf{v}$, a unit vector can be derived by simply dividing the vector by its norm (Theorem 2). This unit vector, called the **normalized vector** of $\mathbf{v}$ is denoted $\hat{\mathbf{v}}$. In a Euclidean vector space, the normalized vector $\hat{\mathbf{v}}$ is the unit vector that points in the same direction as $\mathbf{v}$. Unit vectors are often used to denote a direction in a vector space. That is, since it's magnitude is 1, the relevant information encoded in the vector is the direction in which it points.

Definition 2 *A vector with norm equal to 1 is a* ***unit vector****.*

Theorem 2

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||}$$

Proof:

$$\begin{aligned}
||\hat{\mathbf{v}}|| &= \left|\left|\frac{\mathbf{v}}{||\mathbf{v}||}\right|\right| \\
&= \frac{1}{||\mathbf{v}||}\,||\mathbf{v}|| \\
&= 1
\end{aligned}$$

□