Bounds, supremums, infimums, maximums, and minimums

Upper and lower bounds

An upper bound of a partially ordered set is an element for which all elements in the set are less than that value. This value need not be in the set itself. Similarly, a lower bound of a set is an element for which all elements in the set are greater than that value.

Definition 1 *A set $A \subset B$ is* ***bounded above*** *if there exists an* ***upper bound*** *$b \in B$ such that $a \leq b$ for all $a \in A$.*

Definition 2 *A set $A \subset B$ is* ***bounded below*** *if there exists a* ***lower bound*** *$l \in B$ such that $a \geq l$ for all $a \in A$.*

Supremum

The **supremum** of a subset $A \subset B$ is the least upper bound of that set. This means that if A is bounded above then all upper bounds are greater than or equal to the supremum.

Definition 3 *The* ***supremum*** *of a set $A \subset B$ is an element $s \in B$ such that*

1. *s is an upper bound of A*
2. *if b is an upper bound of A, then $s \leq b$*

We denote the supremum of A as

$$\sup A$$

Theorem 1 provides an alternative definition for the supremum. This definition states that the supremum s is an upper bound of A such that any value less than s is not an upper bound because such a value would be smaller than some element $a \in A$. Theorem 1 proves that this definition is equivalent to the original definition.

Theorem 1 *Assume s is an upper bound for a set A. Then,*

$$s = \sup A \iff \forall \epsilon > 0, \exists\, a \in A \text{ s.t. } s - \epsilon < a$$

Proof:

We first prove the $\implies$ direction:

Let us assume $s = \sup A$. If we pick any positive valued ϵ, the value $s - \epsilon$ is definitely not an upper bound for A because the original definition states that no upper bound is smaller than s. By the definition of an upper bound, this means that there exists a value $a \in A$ where $a > s$.

We now prove the $\impliedby$ direction:

We assume that no matter what positive ϵ we choose, the value $s - \epsilon$ is not an upper bound for A because there is an element in A that is greater than $s - \epsilon$. Thus, there is no value smaller than s that is an upper bound of A, which is exactly how the supremum is defined in Definition 3.

□

Infimum

The infimum of a set is the **greatest lower bound**. That is, if a set is bounded below, all lower bounds are at most as large as the infimum.

Definition 4 *The **infimum** of a subset $A \subset B$ is an element of $s \in A$ such that*

1. *s is a lower bound of A*
2. *if l is a lower bound of A, then $s \geq l$*

We denote the infimum of A as,

$$s = \inf A$$

An alternative definition for the infimum is similar to the alternate definition of the supremum. This definition states that the infimum s is a lower bound such that any value greater than s is not a lower bound because such a value would be greater than some element in A. The proof that this is equivalent to the original definition of the infimum follows the same pattern as the proof of the alternate definition of the supremum.

Theorem 2 *Assume s is a lower bound for a set A. Then,*

$$s = \inf A \iff \forall \epsilon > 0, \text{ there exists an element } a \in A \text{ such that } s + \epsilon > a$$

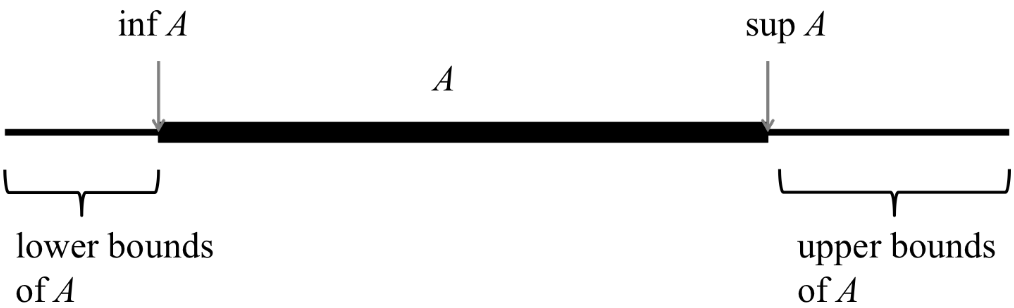

Figure 1: Visualizing the infimum and supremum of a set A.

Maximum and minimum

A maximum is a special type of supremum: it belongs to the set.

Definition 5 *The **maximum** of a set A is an element $m \in A$ such that m is also the supremum.*

Note that not all sets that are bounded above have maximums. For example, consider the set

$$A = \left\{ \frac{n}{n+1} \mid n \in \mathbb{Z} \right\}$$

We see that we can construct an ever larger element in A using ever larger integers $n \in \mathbb{Z}$. These ever larger elements keep getting closer to 1, but never actually reach it. Thus, no matter what value we pick that is less than 1, we can construct a number larger than it, however, we cannot construct 1. Thus, we see that $\sup A = 1$, but $1 \notin A$.

Definition 6 *The **minimum** of a set A is an element $m \in A$ such that m is also the infimum.*

Note that not all sets that are bounded below have minimums. For example, consider the set

$$A = \left\{ \frac{1}{n} \mid n \in \mathbb{Z} \right\}$$

We see that we can construct an ever smaller element in A using ever larger integers $n \in \mathbb{Z}$. These ever smaller elements keep getting closer to 0, but never actually reach it. Thus, no matter what value we pick that is greater than 0, we can construct a number smaller than it, however, we cannot construct 0. Thus, we see that $\inf A = 0$, but $0 \notin A$.

